REAL HYPERSURFACES CONTAINED IN ABELIAN VARIETIES

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1. In a recent note of these Proceedings (April, 1919), I showed that an abelian variety of genus p and rank one, V_p, is birationally transformable into a real one if and only if it possesses $2p$ independent linear cycles $\gamma_1, \gamma_2, \ldots \gamma_{2p}$, with respect to which p integrals of the first kind have a period matrix of type $\Omega = \| \omega_{h,1}, \ldots \omega_{h,p}; i\omega_{h,p+1}, \ldots i\omega_{h,2p} \|$; $(h = 1, 2, \ldots p)$, the (ω)'s being real. I propose now to investigate the number ρ' of algebraically distinct real hypersurfaces which V_p, if real, may have. This number $\rho' \leqq \rho$, Picard number of V_p, may also be defined as the maximum number of real hypersurfaces which cannot be logarithmic singularities of a simple integral of the third kind.

2. In a general way V_p be an abelian variety of rank one, real or not, with the independent linear cycles $\gamma_1, \gamma_2, \ldots \gamma_{2p}$. By associating γ_μ with γ_ν we obtain a superficial cycle (μ, ν) and any other depends upon those of this type. In particular denoting by (A^{p-1}) the two dimensional cycle formed by A^{p-1}, curve of intersection of $p - 1$ algebraic hypersurfaces of the same continuous system as a given one A, we have

$$m (A^{p-1}) \frown \sum_1^{2p} \mu, \nu \; m_{\mu,\nu} (\mu, \nu), \; (m_{\mu,\nu} \text{ integer} = - m_{\nu,\mu}). \tag{1}$$

It may be shown that if no integral of the first kind is constant on A the alternate form

$$\Sigma \; m_{\mu,\nu} \, x_\mu \, y_\nu \tag{2}$$

is a principal form of Ω as defined by Scorza (Palermo Rendic., 1916), and conversely to a principal form (2) corresponds an algebraic hypersurface A. Moreover to algebraically distinct hyersurfaces correspond linearly independent principal forms from which follows at once $\rho = 1 + k$, where k is Scorza's index of singularity for Ω.

3. Let us now assume V_p real. A real hypersurface A of V_p is transformed into itself by T, transformation of the variety which permutes its pairs of conjugate points and this property is characteristic for A. It may be shown that there are real curves A^{p-1};—let the one of No. 2 be one of them, and α its real line (locus of its real points). A small oriented circuit tangent to α in (A^{p-1}) is transformed by T into one of opposite orientation, for in the neighborhood of α, T behaves like an ordinary plane symmetry. It follows that T transforms the superficial cycle (A^{p-1}) into its opposite. Taking into account the fact that this cycle is a two sided manifold and also

the effect of T upon the linear cycles γ_μ of No. 1, we find at once that all the m's not of the type $m'_{\mu, p+\nu}$, $(\mu, \nu \leqq p)$ are equal to zero, hence ρ' *is equal to the number of independent forms of type.*

$$\sum_{1}^{p} {}^{\mu, \nu} \, m'_{\mu, p+\nu} \, (x_\mu y_{p+\nu} - x_{p+\nu} y_\mu) \tag{3}$$

which belong to Ω.

If V_p is pure $\rho' \leqq p$, for otherwise Ω would possess a degenerate form (3). This is to be contrasted with Scorza's result $1 + k \leqq 2p - 1$, or $\rho \leqq 2p - 1$ if Ω is pure.

4. Assuming $\rho' = 2$ let L, L', be the matrices formed by the determinants of two forms (3). They are both of type

$$\left\| \begin{array}{c|c} 0 & \Delta \\ \hline \Delta' & 0 \end{array} \right\|$$

where each square represents a matrix with p rows and columns, the matrices in the main diagonal having only zeroes for terms. As $L^{-1} L'$ is of the form

$$\left\| \begin{array}{c|c} \Delta & 0 \\ \hline 0 & \Delta' \end{array} \right\| = \| \, a_{\mu\nu} \, \|, \, (\mu, \nu = 1, 2, \ldots 2p; \, a_{\mu, p+\nu} = a_{p+\mu, \nu} = 0),$$

V_p has a complex multiplication defined by

$$\sum_{1}^{p} {}^{k} \lambda_{jk} \omega_{k\mu} = \sum_{1}^{p} {}^{\nu} a_{\mu\nu} \omega_{j\nu}; \quad \sum_{1}^{p} {}^{j} \lambda_{j,k} \, \omega_{k, p+\mu} = \sum_{1}^{p} {}^{\nu} a_{p+\mu, p+\nu} \omega_{j, p+\nu}$$

$$(j, \mu = 1, 2, \ldots p),$$

the (λ)'s being necessarily real as they can be replaced by their conjugates. Finally the characteristic equation of this complex multiplication

$$\| \, a_{\mu\nu} - \epsilon_{\mu\nu} x \, \| = 0, \, (\epsilon_{\mu\nu} = 0, \mu \, \nu; \epsilon_{\mu\mu} = 1)$$

is necessarily reducible and a perfect square if V_p is pure.

5. Let us examine the case of a real hyperelliptic surface of rank one. The number ρ' has then the value 1 or 2, if the surface is pure not elliptic. A fundamental period matrix corresponding to linear cycles forming a minimum base may be reduced to the form

$$\left\| \begin{array}{cccc} 1, & 0, & \dfrac{m}{2} + ia, & \dfrac{n}{2} + ib \\[2mm] 0, & 1/\delta, & \dfrac{n}{2} + ib, & \dfrac{r}{2} + ic \end{array} \right\|,$$

where m, n, r, δ, are positive integers and $ac - b^2 > 0$. If $\gamma'_1, \gamma'_2, \ldots \gamma'_{2p}$, are the corresponding linear cycles those of No. 1 are given by

$$\gamma_1 = \gamma'_1, \, \gamma_2 = \gamma'_2, \, \gamma_3 = 2\gamma'_3 - m\gamma'_1 - n\delta\gamma'_2, \, \gamma_4 = 2\gamma'_4 - n\gamma'_1 - r\delta\gamma'_2,$$

and in general $\rho = \rho' = 1$, unless there is a singular relation as defined by G. Humbert. If the surface is not elliptic this relation can only be of type

$$\lambda a + \mu b + \gamma \delta c = 0, \quad (\lambda,\ \mu,\ \nu,\ \text{integers}) \tag{3}$$

and there can only be one such relation. In this case $\rho = \rho' = 2$, and the condition of existence becomes now, assuming as we may, $\nu > 0$,

$$\lambda a^2 + \mu ab - \nu \delta b^2 > 0$$

which assures us of the effective existence of the surface. If there are two singular relations such as (5) the surface is elliptic and $\rho = \rho' = 3$.

In addition to (5) there may be in the non-elliptic case as well as in the other a singular relation independent of (5) and reducible to the form

$$\lambda\ (b^2 - ac) + \mu = 0, \quad (\lambda,\ \mu,\ \text{positive integers})$$

and then $\rho - \rho' = 1$, both cases being realizable. Thus there are six distinct types of real hyperelliptic surfaces for which ρ, ρ' have the values: (1, 1), (1, 2), (2, 2), (2, 3), (3, 3), (3, 4), the last three corresponding to elliptic cases.